Configuration Space and Unitary Representations of the Group of Diffeomorphisms

AUTHOR(S):

OBATA, Nobuaki

数理解析研究所講究録
615巻 1987年 129-153

Configuration Space and Unitary Representations of the Group of Diffeomorphisms

by Nobuaki OBATA (Nagoya Univ.)
尾畑伸明（名大·理）

Introduction

Let X be a connected C^{∞}-manifold and $\mathrm{Diff}_c(X)$ the group of all diffeomorphisms of X which are identical outside a compact set. The group $\mathrm{Diff}_c(X)$, furnished with a natural topology (see §4), becomes an *infinite dimensional Lie group*. Unitary representations of $\mathrm{Diff}_c(X)$ have been studied by many authors [1]-[12]. In this note, we shall consider them from measure-theoretical point of view.

Let Ω be a measurable space on which the group $\mathrm{Diff}_c(X)$ acts as measurable transformations and μ a σ-finite measure on Ω quasi-invariant under $\mathrm{Diff}_c(X)$. Then, the triple $(\Omega, \mu, \mathrm{Diff}_c(X))$ (sometimes abbreviated to (Ω, μ)) is called a *dynamical system* after Kirillov [15]. Given a dynamical system (Ω, μ), we form a unitary representation U of $\mathrm{Diff}_c(X)$:

$$(U(g)f)(\omega) = \left[\frac{d\mu(g^{-1}\omega)}{d\mu(\omega)}\right]^{1/2} A(g,\omega) f(g^{-1}\omega) \quad ,$$

where f is a square-integrable (w. r. t. μ) function on Ω with values in a separable Hilbert space H and $A(g,\omega)$ a 1-cocycle with values in the group of all unitary operators on H. Among many candidates for a dynamical system (Ω, μ), the configuration space (for definition, see §1) seems very interesting in connection with theory of random fields, statistical models (e.g. [16], [18]) and representations of the infinite symmetric group.

In §§1-3, we shall develop a general theory of the configuration space and probability measures on it. In §§4-6, unitary reprsentations of $\mathrm{Diff}_c(X)$ associated with the configuration space will be considered mainly after [12]. Finally, §7 contains a few remarks on another dynamical systems.

§1. Configuration space

Let X be a second countable locally compact space (always assumed Hausdorff). A locally finite subset of X is called a *configuration* in X. The set of all configurations in X will be denoted by $\Omega = \Omega_X$. For any Borel subset B of X, we put

$$\Omega_B = \{ \omega \in \Omega \ ; \ |\omega \cap B^c| = 0 \}$$

and for each integer $n = 0, 1, 2, \cdots$,

$$\Omega^n(B) = \{ \omega \in \Omega \ ; \ |\omega \cap B^c| = 0 \text{ and } |\omega \cap B| = n \},$$

where $|\cdot|$ denotes the cardinality. Note that $\Omega^0(B) = \{\phi\}$, where ϕ is the empty configuration. We set

$$\Omega_f(B) = \bigcup_{n=0}^{\infty} \Omega^n(B) .$$

If B has a compact closure, obviously $\Omega_f(B) = \Omega_B$. For each integer $n = 1, 2, \cdots$, put

$$B^{[n]} = \{ x = (x_1, \cdots, x_n) \in B^n \ ; \ x_i \neq x_j \text{ if } i \neq j \}.$$

The symmetric group $\mathfrak{S}_n$ acts on $B^{[n]}$ as coordinate permutations:

$$x = (x_1, \cdots, x_n) \longmapsto x\sigma = (x_{\sigma(1)}, \cdots, x_{\sigma(n)}) ,$$

where $x \in B^{[n]}$ and $\sigma \in \mathfrak{S}_n$. We tacitly understand $B^{[0]} = \{\phi\}$ and $\mathfrak{S}_0 = \{e\}$. The quotient space $B^{[n]}/\mathfrak{S}_n$ is identified with $\Omega^n(B)$ in an obvious way. We denote by p_B^n the canonical projection $B^{[n]} \longrightarrow \Omega^n(B)$.

We now recall that a subset Y of X becomes a locally compact space with respect to the relative topology if and only if it is locally closed. If $Y \subset X$ is locally closed, $\Omega_f(Y)$ becomes a second countable locally compact space in a natural manner. Furnished with the topological σ-field, $\Omega_f(Y)$ becomes a standard measurable space.

Lemma 1.1. The σ-field of $\Omega_f(Y)$, Y being a locally closed subset of X, is generated by all sets of the form $\{ \omega \in \Omega_f(Y);$ $|\omega \cap B| = n \}$, where B runs over all Borel sets of Y and n all non-negative integers.

Lemma 1.2. Let Y and Y' be two locally closed subsets of X such that $Y \subset Y'$. Then the natural projection $\pi_{YY'}: \Omega_f(Y') \longrightarrow \Omega_f(Y)$ defined by $\pi_{YY'}(\omega) = \omega \cap Y$ is measurable.

These results are not hard to prove. Unless X is a discrete space, the natural projection $\pi_{YY'}$ is *not* continuous in general. Identifying Ω with the projective limit measurable space $\varprojlim \Omega_f(Y)$, we introduce a σ-field in Ω . The measurable space Ω will be called the *configuration space*. The following properties can be shown with the help of general theory of measurable spaces, e.g. [13],[20].

Proposition 1.3. The σ-field of Ω is generated by all sets of the form $\{ \omega \in \Omega ; |\omega \cap B| = n \}$, where B runs over all Borel subsets with compact closures and n all non-negative integers. Moreover, the set $\{ \omega \in \Omega ; |\omega \cap B| = n \}$ is measurable for any Borel set $B \subset X$.

It follows from Lemma 1.3 that $\Omega_B = \{ \omega \in \Omega ; |\omega \cap B^c| = 0 \}$ is measurable for any Borel subset $B \subset X$. We introduce the relative σ-field in it. If $Y \subset X$ is a locally closed set with compact closure, two σ-fields of $\Omega_f(Y)$, i.e. the topological σ-field and the relative σ-field, coincide by Lemma 1.1 and Proposition 1.3.

A partition $\{B_j\}$ of X, B_j being a Borel subset of X, is called *locally finite* if for any compact set $C \subset X$, the number of j's such that $B_j \cap C$ is not empty is finite. The following result means that the configuration space Ω is *infinitely divisible.*

Proposition 1.4. Let $B_1, B_2, \cdots$ be mutually disjoint Borel subsets of X and put $B = \cup B_j$. If $\{B_j\} \cup \{B^c\}$ is a locally finite partition of X, Ω_B is Borel isomorphic to the product space $\Pi\, \Omega_{B_j}$.

This implies that the canonical projection $\pi_{BB'}: \Omega_{B'} \longrightarrow \Omega_B$ is measurable for any two Borel subsets $B \subset B' \subset X$.

Proposition 1.5. The configuration space Ω is Borel isomorphic to the projective limit $\varprojlim \Omega_B$, where B runs over all Borel subsets of X with compact closures. If $B_1 \subset B_2 \subset \cdots \subset X$ be a sequence of Borel subsets with compact closures such that $X = \cup B_j$, then Ω is also Borel isomorphic to $\varprojlim \Omega_{B_j}$.

Proposition 1.6. If $B \subset X$ is a Borel subset with compact closure, Ω_B is a standard measurable space.

§2. Construction of measures

As is well known, every probability measure on the configuration space $\Omega = \varprojlim \Omega_B$ is uniquely determined by a *consistent family of probability measures* $\{\mu_B\}$, where μ_B is a probability measure on Ω_B satisfying $\pi_{BB'}\mu_{B'} = \mu_B$ for all $B \subset B'$.

In this note, by a measure on X we always mean a Borel measure m, possibly $m(X) = \infty$, such that (i) m is non-atomic;

(ii) $m(C) < \infty$ for any compact subset $C \subset X$. For any Borel set $B \subset X$, we denote by m_B^n the restriction of m^n to $B^{[n]}$. We note that $m^n(B^n - B^{[n]}) = 0$. If $B \subset X$ is a Borel subset with compact closure, a probability measure $\exp(m_B)$ on Ω_B is defined by the formula:

$$\exp(m_B) = e^{-m(B)} \sum_{n=0}^{\infty} \frac{1}{n!} p_B^n \, m_B^n ,$$

according to the decomposition $\Omega_B = \cup \, \Omega^n(B)$.

We are interested in probability measures $\mu = \varprojlim \mu_B$ having the property:

(A) μ_B is absolutely continuous with respect to $\exp(m_B)$ for any Borel set $B \subset X$ with compact closure.

This looks rather strong but quite natural for our purpose, (see §4). Given a probability measure μ on Ω with the condition (A), we have a *family of density functions* $\{\rho_B^n\}$ defined by the formula:

$$\mu_B = \sum_{n=0}^{\infty} \frac{1}{n!} p_B^n \left[\rho_B^n \, m_B^n\right]$$

The following properties are satisfied:

(ρ-1) ρ_B^n is a non-negative function on B^n,

(ρ-2) $\rho_B^n(x_1, \cdots, x_n) = \rho_B^n(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$, $\sigma \in \mathfrak{S}_n$,

(ρ-3) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} \int_{B^n} \rho_B^n(x_1, \cdots, x_n) \, dm(x_1) \cdots dm(x_n) = 1$,

(ρ-4) if $B \subset B'$, $\rho_B^{\ell}(x_1, \cdots, x_{\ell}) =$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(B'-B)^n} \rho_{B'}^{\ell+n}(x_1, \cdots, x_{\ell}, x_{\ell+1}, \cdots, x_{\ell+n}) \, dm(x_{\ell+1}) \cdots dm(x_{\ell+n}).$$

The converse is also true, i.e. $\{\rho_B^n\}$ satisfying the above conditions (ρ-1)-(ρ-4) forms a probability measure μ on Ω with (A).

Every probability measure μ on Ω is uniquely determined if the values $\mu(\omega \in \Omega\ ;\ |\omega \cap B_j| = k_j\ ,\ 1 \leqslant j \leqslant N)$ are given, where $B_1, \cdots, B_N \subset X$ are mutually disjoint Borel subsets with compact closures and $k_1, \cdots, k_N$ non-negative integers.

Proposition 2.1. Let μ be a probability measure on Ω satisfying (A). Then, for any mutually disjoint Borel subsets $B_1, \cdots, B_N \subset X$ with compact closures and for any non-negetive integers $k_1, \cdots, k_N$,

$$\mu\left[\ \omega \in \Omega\ ;\ |\omega \cap B_j| = k_j\ ,\ 1 \leqslant j \leqslant N\ \right] =$$

$$= \frac{1}{k_1! \cdots k_N!} \int_{B_1^{k_1} \times \cdots \times B_N^{k_N}} \rho_B^k(x_1, \cdots, x_k)\ dm(x_1) \cdots dm(x_k)\ ,$$

where $B = \cup B_j$ and $k = \sum k_j$.

Generally speaking, it seems difficult to find an explicit description of a family of density functions $\{\rho_B^n\}$. A particular case will be considered in the next section.

§3. A characterization of certain probability measures

In this section we shall consider the case when every density function ρ_B^n reduces a constant function, say c_B^n . The conditions $(\rho\text{-}1)$-$(\rho\text{-}4)$ become simpler as follows:

$$(c\text{-}1)\quad c_B^n \geqslant 0\ ,$$

$$(c\text{-}2)\quad \sum_{n=0}^{\infty} \frac{c_B^n}{n!} (m(B))^n = 1\ ,$$

$$(c\text{-}3)\quad \sum_{n=0}^{\infty} \frac{c_B^{\ell+n}}{n!} (m(B'-B))^n = c_B^{\ell} \qquad \text{for } B \subset B'\ .$$

We shall find a lucid expression of $\{c_B^n\}$.

We define a holomorphic function $h_B(t)$ by the formula:

$$h_B(t) = \sum_{n=0}^{\infty} \frac{c_B^n}{n!} t^n \quad , \qquad |t| < m(B) \ .$$

The conditions (c-1)-(c-3) are replaced with the following

(h-1) $h_B^{(n)}(0) \geqslant 0$, $n \in \mathbb{N}$,

(h-2) $h_B(m(B)) = 1$,

(h-3) $h_{B'}^{(n)}(m(B'-B)) = h_B^{(n)}(0)$, whenever $B \subset B'$.

We note that $c_B^n = h_B^{(n)}(0)$. It follows from (h-3) that

$$\sum_{n=0}^{\infty} \frac{h_{B'}^{(n)}(m(B'-B))}{n!} (t-m(B'-B))^n = \sum_{n=0}^{\infty} \frac{h_B^{(n)}(0)}{n!} (t-m(B'-B))^n \ .$$

This implies that $h_{B'}(t+m(B')) = h_B(t+m(B))$. Thus, by analytic continuation, we get a holomorphic function $H(t)$ in $D \subset \mathbb{C}$ such that

$$H(t) = h_B(t+m(B)) \qquad \text{if} \quad |t+m(B)| < m(B).$$

Here $D = \{ \ |t+m(X)|< m(X) \}$ or $\{ \ \mathrm{Re}(t)<0 \ \}$ according as $m(X) < \infty$ or $m(X) = \infty$. The following assertion is then direct.

Lemma 3.1. The function $H(t)$ has the following conditions:

(H-1) $H(t)$ is holomorphic in D,

(H-2) $H^{(n)}(t) \geqslant 0$ if $-m(X) < t < 0$,

(H-3) $H(0) = 1$.

Conversely, if a function $H(t)$ enjoys the conditions (H-1)-(H-3) above, $\{ \ c_B^n = H^{(n)}(-m(B)) \ \}$ satisfies the conditions (c-1)-(c-3).

Thus, if we are given a measure m on X and a function $H(t)$ with the condition (H), where (H) stands for the conditions (H-1)-(H-3), a probability measure on Ω is constructed and denoted by $\mu_{m,H}$.

Summing up the above results, we have

Theorem 3.2. Let μ be a probability measure on Ω which satisfies the condition (A). Then, every density function ρ_B^n reduces a constant function if and only if $\mu = \mu_{m,H}$ for some function $H(t)$ with the condition (H). In this case $\rho_B^n \equiv H^{(n)}(-m(B))$.

Example. As is easily verified, $H(t) = e^t$ satisfies the condition (H). In other words, $\{\exp(m_B)\}$ is a consistent family of probability measures. The probability measure $\mu_{m,H} = \varprojlim \exp(m_B)$ is called the *Poisson measure* on Ω and denoted by $\exp(m)$.

Remark. Even if a probability measure μ on Ω satisfies the condition (A), it is *not* necessarily absolutely continuous with respect to the Poisson measure $\exp(m)$.

The following result means that the Poisson measure is *infinitely divisible*. The proof is easy and omitted.

Proposition 3.3. Let $\{B_j\}$ be a locally finite partition of X and μ_j the image measure of the Poisson measure $\exp(m)$ under the canonical projection $\Omega \longrightarrow \Omega_{B_j}$. Then, $\exp(m) = \Pi\, \mu_j$ according to $\Omega = \Pi\, \Omega_{B_j}$.

We shall now give a decomposition of the measures $\mu_{m,H}$. If the measure m on X is finite, i.e. $m(X) < \infty$, the polynomial $H_n(t)$ given by

$$H_n(t) = \left[1 + \frac{t}{m(X)} \right]^n \quad , \; n = 0, 1, 2 \cdots,$$

also satisfies the condition (H). The corresponding probability measure is denoted by $\mu_{m,n}$ for simplicity. Then we have the following

Proposition 3.4. The measure $\mu_{m,n}$ is concentrated on $\Omega^n(X)$ ($\subset \Omega$). Moreover, the restriction of $\mu_{m,n}$ to $\Omega^n(X)$ coincides with the image measure of $(m(X))^{-n}\, m^n$ under the canonical projection $p_X^n : X^{[n]} \longrightarrow \Omega^n(X)$.

Obviously, $H(t) = e^{ct}$, $c \geqslant 0$, satisfies the condition (H) and that the corresponding probability measure $\mu_{m,H}$ is the Poisson measure $\exp(cm)$. Here we included a Dirac measure concentrated at ϕ (the empty configuration) as a Poisson measure $\exp(0)$. With these preparations, we can now state the following

Theorem 3.5. Let $\mu_{m,H}$ be a probability measure on Ω corresponding to a measure m on X and a function H(t) with (H).

(1) If $m(X) < \infty$, there exists a unique sequence $\lambda_0, \lambda_1, \cdots \geqslant 0$ with $\sum_{n=0}^{\infty} \lambda_n = 1$ such that $\mu_{m,H} = \sum_{n=0}^{\infty} \lambda_n\, \mu_{m,n}$.

(2) If $m(X) = \infty$, there exists a unique Borel probability measure λ on $[0,\infty)$ such that $\mu_{m,H} = \int_{[0,\infty)} \exp(cm)\, d\lambda(c)$.

Proof. It follows from (H-2) that H(t) is a totally monotonic function on the interval $(-m(X), 0)$. Then, by virtue of the Bernstein's theorem (e.g. [14]), we have

(1) If $m(X) < \infty$, there exists a unique sequence $\lambda_0, \lambda_1, \cdots \geqslant 0$ with $\sum_{n=0}^{\infty} \lambda_n = 1$ such that $\mu_{m,H} = \sum_{n=0}^{\infty} \lambda_n\, \mu_{m,n}$.

(2) If $m(X) = \infty$, there exists a unique Borel probability

measure λ on $[0,\infty)$ such that $H(t) = \int_{[0,\infty)} e^{ct}\, d\lambda(c)$.

Here we consider only (2) in order to avoid repeating almost the same argument twice. Put

$$\mu' = \int_{[0,\infty)} \exp(cm)\, d\lambda(c) .$$

We have only to show that $\pi_B\, \mu' = \pi_B\, \mu_{m,H}$ for any Borel subset $B \subset X$ with compact closure. In fact,

$$\begin{aligned}\pi_B\, \mu' &= \int_{[0,\infty)} \pi_B \exp(cm)\, d\lambda(c) \\ &= \int_{[0,\infty)} \sum_{n=0}^{\infty} \frac{e^{-cm(B)}\, c^n}{n!}\, p_B^n\, m_B^n \quad d\lambda(c) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{[0,\infty)} e^{-cm(B)} c^n\, d\lambda(c) \right) p_B^n\, m_B^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} H^{(n)}(-m(B))\, p_B^n\, m_B^n \;=\; \pi_B\, \mu_{m,H} .\end{aligned}$$

This completes the proof. Q. E. D.

Remark. If $m(X) < \infty$, the Poisson measure $\exp(m)$ itself is decomposed as follows:

$$\exp(m) = \sum_{n=0}^{\infty} e^{-m(X)} \frac{m(X)^n}{n!}\, \mu_{m,n} .$$

Lemma 3.6. Let $B_1, \cdots, B_N$ be mutually disjoint Borel subsets of X with compact closures and $k_1, \cdots, k_N$ non-negative integers. Then

$$\mu_{m,H}\left(\omega \in \Omega \;;\; |\omega \cap B_j| = k_j \;,\; 1 \leqslant j \leqslant N \right) =$$

$$= H^{(k)}(-m(B)) \prod_{j=1}^{N} \frac{m(B_j)^{k_j}}{k_j!} \quad ,$$

where $B = \cup B_j$ and $k = \Sigma k_j$.

This is immediate from Proposition 2.1. Then we have the following result.

Proposition 3.7. Let $B \subset X$ be an arbitrary Borel subset and n a non-negative integer. Then

$$\mu_{m,H}\left[\omega \in \Omega \ ; \ |\omega \cap B| = n\right] = \begin{cases} H^{(n)}(-m(B))\ \dfrac{m(B)^n}{n!} & , \text{ if } m(B) < \infty, \\ 0 & , \text{ othewise }. \end{cases}$$

In particular, if $m(X) = \infty$, the Poisson measure $\exp(m)$ is concentrated on the set of all infinite configurations, namely, $\exp(m)(\Omega - \Omega_f(X)) = 1$.

§4. Quasi-invariant measures on the configuration space

From this section on, X denotes a connected orientable (for technical simplicity) C^∞-manifold with a C^∞-volume form m. We always define a measure on X by the volume form and denote it by the same symbol. Let $\mathrm{Diff}_c(X)$ the group of all diffeomorphisms of X which are identical outside a compact set (depending on $g \in \mathrm{Diff}_c(X)$). We introduce a topology in $\mathrm{Diff}_c(X)$ as follows: The convergence of $g_n \longrightarrow g$ (as $n \longrightarrow \infty$) signifies that g and all g_n are identical outside a fixed compact set and that $g_n(x) \longrightarrow g(x)$ with all the derivatives uniformly in X. The group $\mathrm{Diff}_c(X)$ becomes a topological group (actually infinite dimensional Lie group). We denote by $\mathrm{Diff}_c(X,m)$ the subgroup of all diffeomorphisms in $\mathrm{Diff}_c(X)$ which preserve the volume form m.

The group $\mathrm{Diff}_c(X)$ acts on the configuration space Ω by means of the maps:

$$\omega = \{x_1, x_2, \cdots\} \longmapsto g\omega = \{g(x_1),\ g(x_2),\ \cdots\}\ ,\ \omega \in \Omega.$$

Obviously, each $\Omega^n(X)$ is stable under this action.

Recall that every non-zero σ-finite Borel measure on $\mathbb{R}^n$ which is quasi-invariant under translations is equivalent to the Lebesgue measure (e.g. [20]). Then we can prove the following

Proposition 4.1. Every non-zero σ-finite Borel measure on $\Omega^n(X)$ which is quasi-invariant under $\mathrm{Diff}_c(X)$ is equivalent to the image measure $p_X^n\, m^n$, where $p_X^n : X^{[n]} \longrightarrow \Omega^n(X)$ is the canonical projection.

Proposition 4.2. If a probability measure μ on Ω is quasi-invariant under the action of $\mathrm{Diff}_c(X)$, it enjoys the property (A), namely, for any Borel subset $B \subset X$ with compact closure, μ_B is absolutely continuous with respect to $\exp(m_B)$.

Proof. Fix a sequence of open sets with compact closures $X_1 \subset X_2 \subset \cdots$ such that $X = \bigcup X_j$. It follows from Proposition 1.5 that $\Omega = \varprojlim \Omega_{X_j}$. Suppose that we are given a quasi-invariant measure μ on Ω. We write $\mu_j = \mu_{X_j}$ for brevity. We define a measure μ_j^n on $\Omega^n(X)$ by

$$\mu_j = \sum_{n=0}^{\infty} \frac{1}{n!}\, \mu_j^n \quad , \text{ according as } \quad \Omega_{X_j} = \bigcup_{n=0}^{\infty} \Omega^n(X_j) \ .$$

By assumption we see that μ_j^n is quasi-invariant under $\mathrm{Diff}_c(X_j)$. It follows from Proposition 4.1 that there exists a measurable function $\rho_{X_j}^n(x_1, \cdots, x_n)$ such that

(i) $\rho_{X_j}^n(x_1, \cdots, x_n) > 0 \quad \text{or} \quad \equiv 0$,

(ii) $\rho_{X_j}^n(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) = \rho_{X_j}^n(x_1, \cdots, x_n)$ for any $\sigma \in \mathfrak{S}_n$,

(iii) $\mu_j^n = p_{X_j}^n \left[\rho_{X_j}^n\, m_{X_j}^n \right]$

For any Borel subset $B \subset X$ with compact closure, we define ρ_B^ℓ by the formula (ρ-4) in §2 (taking B' = X_j). Then,

$$\mu_B = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} p_B^{\ell} \left[\rho_B^{\ell} \, m_B^{\ell} \right] .$$

Hence μ_B is absolutely continuous with respect to $\exp(m_B)$. Q.E.D.

The following result is a direct consequence of a general theory ([20]).

Proposition 4.3. Let $X_1 \subset X_2 \subset \cdots$ be a sequence of open subsets with compact closures such that $X = \cup X_j$. A probability measure μ on Ω is quasi-invariant under $\mathrm{Diff}_c(X)$ if and only if (i) $g\mu_j$ is absolutely continuous with respect to μ_j for any $g \in \mathrm{Diff}_c(X_j)$ and $j = 1,2,\cdots$; (ii) the Radon Nikodym derivatives $d(g\mu_j)/d\mu_j$ converges in $L^1(\mu)$. (Here we write $\mu_j = \mu_{X_j}$.)

Applying this result to the measure $\mu_{m,H}$, we obtain

Proposition 4.4. The probability measure $\mu_{m,H}$, where H is a function with the property (H), is quasi-invariant under $\mathrm{Diff}_c(X)$ and invariant under $\mathrm{Diff}_c(X,m)$. Furthermore, we have

$$\frac{d\mu_{m,H}(g^{-1}\omega)}{d\mu_{m,H}(\omega)} = \prod_{x\in\omega} \frac{dm(g^{-1}x)}{dm(x)} .$$

Finally, we can prove the following assertions, c.f. [12].

Proposition 4.5. (1) If $m(X) = \infty$, the Poisson measure $\exp(m)$ is ergodic under $\mathrm{Diff}_c(X)$.

(2) If $m(X) = \infty$ and if $\dim X > 1$, the Poisson measure $\exp(m)$ is ergodic under $\mathrm{Diff}_c(X,m)$.

(3) If $m(X) < \infty$, the probability measure $\mu_{m,n}$ is ergodic under $\mathrm{Diff}_c(X)$ for all $n = 0, 1, 2, \cdots$.

§5. Unitary representations associated with finite configurations

We consider the finite configuration space $\Omega^n(X)$. There is a (essentially) unique measure μ_n which is quasi-invariant under $\mathrm{Diff}_c(X)$ (Proposition 4.1). A map $\sigma : \mathrm{Diff}_c(X) \times \Omega \longrightarrow \mathfrak{S}_n$ is called a *1-cocycle* (of $\mathrm{Diff}_c(X)$ with values in the group of maps $\Omega^n(X) \longrightarrow \mathfrak{S}_n$) if it satisfies

$$\sigma(g_1 g_2, \omega) = \sigma(g_1, \omega)\ \sigma(g_2, g_1^{-1}\omega).$$

Two 1-cocycles σ and σ' are said to be *cohomologuous* if there exists a map $\sigma_0 : \Omega \longrightarrow \mathfrak{S}_n$ such that

$$\sigma'(g, \omega) = \sigma_0(\omega)\ \sigma(g, \omega)\ \sigma_0(g^{-1}\omega)^{-1}.$$

Let (ρ, W^ρ) be a unitary representation of $\mathfrak{S}_n$. We form a unitary representation $\pi^{\rho,\sigma}$ of $\mathrm{Diff}_c(X)$ by the formula:

$$(\pi^{\rho,\sigma}(g) f)(\omega) = \left[\frac{d\mu_n(g^{-1}\omega)}{d\mu_n(\omega)}\right]^{1/2} \rho(\sigma(g, \omega))\ f(g^{-1}\omega),$$

$f \in L^2(\Omega, \mu_n) \otimes W^\rho$, i.e. a square integrable function on Ω with values in W^ρ.

In what follows, we shall restrict ourselves to a particular case when a 1-cocycle σ comes from a measurable cross section. Fix a measurable cross section s for the canonical projection $p_X^n : X^{[n]} \longrightarrow \Omega^n(X)$. For each $g \in \mathrm{Diff}_c(X)$ and $\omega \in \Omega$ there exists a unique permutation $\sigma(g, \omega)$ such that

$$s(g^{-1}\omega) = g^{-1}(s(\omega))\ \sigma(g, \omega).$$

Clearly, $\sigma(g, \omega)$ becomes a 1-cocycle. We note that two 1-cocycles constructed from cross sections are cohomologuous. If σ is such a 1-cocycle, the representation $\pi^{\rho,\sigma}$ will be denoted by π^{ρ}.

The representations π^{ρ} are realized in a slightly different way as follows. The group $\mathrm{Diff}_c(X)$ and the symmetric group $\mathfrak{S}_n$ acts on $X^{[n]}$ by means of the maps:

$$x = (x_1, \cdots, x_n) \longrightarrow gx = (g(x_1), \cdots, g(x_n)), \quad g \in G.$$

$$x = (x_1, \cdots, x_n) \longrightarrow x\sigma = (x_{\sigma(1)}, \cdots, x_{\sigma(n)}), \quad \sigma \in \mathfrak{S}_n .$$

Since we always assume that $X^{[n]}$ is equpped with a measure m^n, where m is a C^{∞}-volume form on X, we have $L^2(X^n) = L^2(X^{[n]})$. For any unitary representation (ρ, W^{ρ}) of $\mathfrak{S}_n$ we put

$$H^{\rho} = \left\{ f \in L^2(X^n, m^n) \otimes W^{\rho} \; ; \; f(x\sigma) = \rho^{-1}(\sigma) f(x), \; x \in X^n, \; \sigma \in \mathfrak{S}_n \right\}.$$

Then the unitary representation π^{ρ} is realized on H^{ρ}:

$$(\pi^{\rho}(g) f)(x) = \left[\prod_{k=1}^{n} \frac{dm(g^{-1}x_k)}{dm(x_k)} \right]^{1/2} f(g^{-1}x), \quad f \in H^{\rho}.$$

The following result was proved in [8] and [12].

Theorem 5.1. If dim X > 1, the representations π^{ρ} of $\mathrm{Diff}_c(X)$ are irreducible and mutually inequivalent.

The representations π^{ρ} arise quite naturally. We now introduce a unitary representation U_n of $\mathrm{Diff}_c(X)$ by the formula:

$$(U_n(g) f)(x) = \left[\prod_{k=1}^{n} \frac{dm(g^{-1}x_k)}{dm(x_k)} \right]^{1/2} f(g^{-1}x), \quad f \in L^2(X^n).$$

If $\sigma \in \mathfrak{S}_n$, we define a unitary operator $V_n(\sigma)$ on $L^2(X^n)$ by

$$(V_n(\sigma)f)(x) = f(x\sigma) \ , \ f \in L^2(X^n) \ .$$

Obviously, $U_n(g)$ and $V_n(\sigma)$ commute each other. The following argument is similar to the representation theory of general linear group. We denote by $\mathfrak{S}_n^\wedge$ the set of all equivalence classes of irreducible unitary representations of $\mathfrak{S}_n$. For each equivalence class of $\mathfrak{S}_n^\wedge$ we fix a representation matrix $\rho = (\rho_{ij})_{1 \leqslant i,j \leqslant \dim \rho}$. We put

$$P^\rho_{ij} = \frac{\dim \rho}{n!} \sum_{\sigma \in \mathfrak{S}_n} \rho_{ij}(\sigma^{-1}) V_n(\sigma) \ , \quad 1 \leqslant i,j \leqslant \dim \rho \ , \ \rho \in \mathfrak{S}_n^\wedge \ .$$

Obviously, these operators commute with $U_n(g)$, for all $g \in \mathrm{Diff}_c(X)$. The following relations are easily verified.

$$P^\rho_{ij} P^{\rho'}_{k\ell} = 0 \quad \text{if } \rho \neq \rho' \ .$$

$$P^\rho_{ij} P^\rho_{k\ell} = \delta_{jk} P^\rho_{i\ell} \ .$$

$$(P^\rho_{ij})^* = P^\rho_{ji} \ .$$

$$\sum_{\rho \in \mathfrak{S}_n^\wedge} \sum_{i=1}^{\dim \rho} P^\rho_{ii} = 1 \ .$$

Then we have the following

Lemma 5.2. P^ρ_{ii} is a non-zero projection. P^ρ_{ij} is a partial isometry with initial projection P^ρ_{jj} and final projection P^ρ_{ii}.

Proof. For proving $P^\rho_{ii} \neq 0$, we let $f(x_1, \cdots, x_n)$ be the indicator function of $\mathcal{O}_1 \times \cdots \times \mathcal{O}_n$, where $\mathcal{O}_1, \cdots, \mathcal{O}_n$ are mutually disjoint open sets of X with compact closures. Then we can show that

$$\sum_{i=1}^{\dim \rho} P^\rho_{ii} f \neq 0 \ .$$

Since P^ρ_{ii} , $1 \leqslant i \leqslant \dim \rho$, are mutually equivalent projections, we have $P^\rho_{ii} \neq 0$. The rest of the assertion is immediate from the relations given above. Q.E.D.

The following assertion is easy to see.

Proposition 5.3. The unitary representation U_n is decomposed into a sum of unitary representations π^ρ:

$$U_n \simeq \sum_{\rho\in\mathfrak{S}_n^\wedge} (\dim \rho)\ \pi^\rho ,$$

according as

$$L^2(X^n, m^n) = \sum_{\rho\in\mathfrak{S}_n^\wedge} \sum_{i=1}^{\dim \rho} P^\rho_{ii}(L^2(X^n, m^n)) .$$

Remark. In case when $\dim X = 1$, i.e. $X = S^1$ (circle) or $X = \mathbb{R}^1$ (real line), the decomposition given in Proposition 5.3 is also valid. In these cases, however, the representation π^ρ is further decomposed. We omit a detailed discussion here.

Remark. Further detailed arguments were done in [8], where the representations π^ρ were obtained in a frame work of *orbit method*.

§6. Unitary representations associated with infinite configurations

Let μ be a probability measure on Ω which is quasi-invariant under $\mathrm{Diff}_c(X)$. In this section we always assume that μ is concentrated on the set of all *infinite* configurations, i.e. $\mu(\Omega-\Omega_f(X)) = 1$. For instance, the Poisson measure $\exp(m)$ is such a measure if $m(X) = \infty$ (see Proposition 3.7). We agree to understand that Ω consists of infinite configurations.

The *infinite symmetric group* is the discrete group of all finite permutations of $\mathbb{N} = \{1, 2, \cdots\}$ and denoted by $\mathfrak{S}_\infty$. A map $\sigma : \mathrm{Diff}_c(X) \times \Omega \longrightarrow \mathfrak{S}_\infty$ is called a *1-cocycle* (of $\mathrm{Diff}_c(X)$ with values in the group of maps $\Omega \longrightarrow \mathfrak{S}_\infty$) if it satisfies

$$\sigma(g_1 g_2, \omega) = \sigma(g_1, \omega)\ \sigma(g_2, g_1^{-1}\omega), \quad \mu\text{-a.e. } \omega .$$

Two 1-cocycles σ and σ' are said to be *cohomologuous* if there exists a map $\sigma_0 : \Omega \longrightarrow \mathfrak{S}_\infty$ such that

$$\sigma'(g, \omega) = \sigma_0(\omega)\ \sigma(g, \omega)\ \sigma_0(g^{-1}\omega)^{-1}, \quad \mu\text{-a.e.}$$

We associate with each unitary representation (π, H^π) of $\mathfrak{S}_\infty$, a unitary representation $U^{\mu,\pi,\sigma}$ of $\mathrm{Diff}_c(X)$:

$$(*) \qquad (U^{\mu,\pi,\sigma}(g)f)(\omega) = \left[\frac{d\mu(g^{-1}\omega)}{d\mu(\omega)}\right]^{1/2} \pi(\sigma(g, \omega))\ f(g^{-1}\omega),$$

where $f \in L^2(\Omega, \mu) \otimes H^\pi$, i.e. a square-integrable function on Ω with values in H^π. For unitary representations of $\mathfrak{S}_\infty$, see [17] and its references. A complete classification of the unitary representations $U^{\mu,\pi,\sigma}$ has not been obtained yet. In what follows, we shall mention several particular cases.

If π is the trivial representation, we write $U^\mu = U^{\mu,\pi,\sigma}$ for simplicity. Then

$$(U^\mu(g)f)(\omega) = \left[\frac{d\mu(g^{-1}\omega)}{d\mu(\omega)}\right]^{1/2} f(g^{-1}\omega), \quad f \in L^2(\Omega, \mu).$$

The following result was proved in [12].

Theorem 6.1. The unitary representation U^μ is irreducible if and only if μ is ergodic under $\mathrm{Diff}_c(X)$.

We have discussed in §3, a class of quasi-invariant probability measures $\mu_{m,H}$. Recall that $\mu_{m,H}$ is concentrated on the set of infinite configurations if $m(X) = \infty$. For simplicity we write $\mu = \mu_{m,H}$. We shall consider the corresponding representation U^μ of

$\mathrm{Diff}_c(X)$. Let $\mathscr{H}$ be a closed subspace of $L^2(\Omega,\mu)$ spanned by $\{ U^\mu(g)\mathbf{1} ;\ g \in \mathrm{Diff}_c(X) \}$, where $\mathbf{1}(\omega) \equiv 1$. We denote by U the restriction of U^μ on the space $\mathscr{H}$, which becomes a cyclic representation of $\mathrm{Diff}_c(X)$ with a cyclic vector $\mathbf{1}$.

Lemma 6.2. We have

$$(U^\mu(g)\mathbf{1},\ \mathbf{1}) = H\left(\int_X \left(\left(\frac{dm(g^{-1}x)}{dm(x)}\right)^{1/2} - 1\right) dm(x)\right), \quad g \in \mathrm{Diff}_c(X).$$

Proof. It follows from Proposition 4.4 that

$$(U^\mu(g)\mathbf{1},\ \mathbf{1}) = \int_\Omega \prod_{x\in\omega} \left(\frac{dm(g^{-1}x)}{dm(x)}\right)^{1/2} d\mu_{m,H}(\omega) .$$

Take a Borel subset $B \subset X$ with compact closure such that $\mathrm{supp}\ g \subset B$. Then the last integral becomes

$$\int_{\Omega_B} \prod_{x\in\omega} \left(\frac{dm(g^{-1}x)}{dm(x)}\right)^{1/2} d\left[\sum_{n=0}^{\infty} \frac{1}{n!} H^{(n)}(-m(B))\ p_B^n\ m_B^n\right](\omega)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} H^{(n)}(-m(B)) \left(\int_B \left(\frac{dm(g^{-1}x)}{dm(x)}\right)^{1/2} dm(x)\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} H^{(n)}(-m(B)) \left(\int_B \left(\left(\frac{dm(g^{-1}x)}{dm(x)}\right)^{1/2} - 1\right) dm(x) + m(B)\right)^n$$

$$= H\left(\int_X \left(\left(\frac{dm(g^{-1}x)}{dm(x)}\right)^{1/2} - 1\right) dm(x)\right),$$

as desired. Q.E.D.

Viewing Proposition 4.5 and Lemma 6.2 we have

Theorem 6.3. Assume that $\dim X > 1$ and $m(X) = \infty$. Then the unitary representation $U^{\exp(m)}$ is of class one with respect to the subgroup $\mathrm{Diff}_c(X,m)$, namely, (i) it is irreducible; (ii) there exists a unique non-zero vector $\mathbf{1}$ (up to constant factor) which is fixed under $\mathrm{Diff}_c(X,m)$. The spherical function is given by

$$(U^{\exp(m)}(g)\mathbf{1},\ \mathbf{1}) = \exp\left[\int_X\left[\left(\frac{dm(g^{-1}x)}{dm(x)}\right)^{1/2} - 1\right]dm(x)\right].$$

In particular, if $c \neq c'$ $(c, c' \geqslant 0)$, two representations $U^{\exp(cm)}$ and $U^{\exp(c'm)}$ are not equivalent.

Remark. The representation $U^{\exp(m)}$ is also realized on $L^2(\mathscr{D}'(X),\nu)$, where ν is the standard Gaussian measure on the space $\mathscr{D}'(X)$ of distributions on X, (see [12]).

With the help of Proposition 6.3 and Theorem 3.5, we obtain the following

Proposition 6.4. The cyclic representation $(U,\mathscr{H},\mathbf{1})$, $\mu = \mu_{m,H}$, is decomposed into a sum of irreducible representations:

$$U^{\mu} \simeq \int_{[0,\infty)}^{\oplus} U^{\exp(cm)}\, d\lambda(c).$$

The unitary representation $\pi_n = \pi^{\mathbf{1}_n}$, where $\mathbf{1}_n$ is the trivial representation of $\mathfrak{S}_n$, is called an *n-particle representation* after [2]. Let $X_1 \subset X_2 \subset \cdots \subset X$ be connected open submanifolds with compact closures such that $X = \bigcup X_n$. Choose a sequence of C^{∞}-functions $\alpha_n(x)$ on X such that

(i) $0 < \alpha_n(x) \leqslant 1$,

(ii) $\alpha_n(x) = 1$ if $x \in X_n$,

(iii) $\int_X \alpha_n(x)\, dm(x) < +\infty$.

Define a new volume form $m_n = \alpha_n(x)m$, $n = 1, 2, \cdots$. Obviously, $m_n(X) < +\infty$ and m_n is equivalent to m. For each $n = 0, 1, 2, \cdots$ we form π_n using the volume form m_n. Then it becomes of class one with respect to $\mathrm{Diff}_c(X, m_n)$. As a normalized fixed vector under $\mathrm{Diff}_c(X, m_n)$ we take $f_n(x) \equiv (m_n(X))^{-n/2}$. The spherical function is given by

$$v_n(g) = (\pi_n(g) f_n, f_n)$$

$$= \left[\frac{1}{m_n(X)} \int_X \left[\frac{dm_n(g^{-1}x)}{dm_n(x)} \right]^{1/2} dm_n(x) \right]^n .$$

The following assertion suggests that the irreducible representation $U^{\exp(m)}$ could be considered as a limit of n-particle representations.

Proposition 6.5. If $\lim_{n\to\infty} \frac{n}{m_n(X)} = c\ (>0)$, we have

$$\lim_{n\to\infty} v_n(g) = (U^{\exp(cm)}(g)\mathbf{1},\ \mathbf{1}) .$$

Proof. For each $g \in \mathrm{Diff}_c(X)$, we take a sufficiently large n such that $\mathrm{supp}\, g \subset X_n$. Then, by definition we have

$$\left[\frac{dm_n(g^{-1}x)}{dm_n(x)}\right]^{1/2} - 1 = \left[\frac{dm(g^{-1}x)}{dm(x)}\right]^{1/2} - 1 , \quad \text{if } x \in X_n ,$$

$$= 0 , \quad \text{otherwise.}$$

Viewing this, we have

$$v_n(g) = \left[1 + \frac{1}{m_n(X)} \int_X \left[\left[\frac{dm(g^{-1}x)}{dm(x)}\right]^{1/2} - 1\right] dm(x) \right]^n$$

$$\longrightarrow \exp\left[c\int_X \left[\left[\frac{dm(g^{-1}x)}{dm(x)}\right]^{1/2} - 1\right] dm(x)\right] \quad \text{as } n \longrightarrow \infty .$$

The assertion is now immediate from Theorem 6.3. Q.E.D.

Finally we consider unitary representations of $\mathrm{Diff}_c(X)$ of the form (*). We consider

$$X^{[\infty]} = \{ x = (x_1, x_2, \cdots) \in X^{\infty} ; x_i \neq x_j \text{ if } i \neq j \},$$

furnished with the relative σ-field. (X^{∞} is furnished with the usual product σ-field.) An injective map $s : \Omega \longrightarrow X^{[\infty]}$, $s(\omega) = (s_1(\omega), s_2(\omega), \cdots)$, is called an *indexing* ([12]) if (i) $\omega = \{ s_1(\omega), s_2(\omega), \cdots \}$; (ii) s is a Borel isomorphism between Ω and $s(\Omega)$. For each $g \in \mathrm{Diff}_c(X)$ and $\omega \in \Omega$, there exists a unique automorphism $\sigma(g,\omega)$ of $\mathbb{N}$ such that

$$s(g^{-1}\omega) = g^{-1}(s(\omega))\ \sigma(g,\omega) .$$

If every $\sigma(g,\omega)$ belongs to $\mathfrak{S}_{\infty}$, the indexing s is called *correct*. In this case, $\sigma(g,\omega)$ is called a *correct 1-cocycle*.

The finite symmetric groups $\mathfrak{S}_n$ are naturally regarded as subgroups of $\mathfrak{S}_{\infty}$. We denote by $\mathfrak{S}_{\infty-n}$ the subgroup of all finite permutations leaving $n+1, n+2, \cdots$ fixed. If ρ is an irreducible representation of $\mathfrak{S}_n$, we write $\rho*\mathbf{1} = \mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_{\infty-n}}^{\mathfrak{S}_{\infty}} \rho\times\mathbf{1}$, where $\mathbf{1}$ is the trivial representation of $\mathfrak{S}_{\infty-n}$. Then we can prove the following

Proposition 6.6. Let σ be a correct 1-cocycle. Then the unitary representation $U^{\exp(m), \rho*\mathbf{1}, \sigma}$ is equivalent to $\pi^{\rho}\otimes U^{\exp(m)}$.

For irreducibility we have the following result ([12]).

Theorem 6.7. Assume that $\dim X > 1$. The tensor product $\pi^{\rho}\otimes U^{\mu}$ is irreducible if and only if μ is ergodic under $\mathrm{Diff}_c(X)$ and ρ is an irreducible representation of $\mathfrak{S}_n$.

§7. Concluding remarks

In [3] and [4], a probability measure is introduced on the space of all closed subsets of X, X being a compact manifold, and corresponding unitary representations are discussed. In particular, an example of a probability measure on the space of all convergent sequences in X is given.

Probability measures can be constructed on the group Γ of all homeomorphisms of the circle which is quasi-invariant under $\mathrm{Diff}(S^1)$. (The action of $\mathrm{Diff}(S^1)$ is defined by $\gamma \longmapsto g\circ\gamma$, $\gamma \in \Gamma$.) For example, see [19]. This suggests a possibility to consider a *regular representation* of $\mathrm{Diff}(S^1)$.

References

[1] G.A.Goldin, Nonrelativistic current algebras as unitary representations of groups, J.Math.Phys., 12 (1971), 462-487.

[2] G.A.Goldin, J.Grodnik, R.T.Powers and D.H.Sharp, Nonrelativistic current algebra in the N/V limit, J.Math.Phys., 15 (1974), 88-100.

[3] R.S.Ismagilov, Unitary representations of the group of diffeomorphisms of a circle, Funct.Anal.Appl., 5 (1971), 209-21

[4] R.S.Ismagilov, On unitary representations of the group of the diffeomorphisms of a compact manifold, Math.USSR,Izv., 6 (1972) 181-209.

[5] R.S.Ismagilov, Unitary representations of group of diffeomorphisms of space $\mathbf{R}^n$, $n \geqslant 2$, Funct.Anal.Appl., 9 (1975), 154-155.

[6] R.S.Ismagilov, Unitary representations of groups of measure-

preserving diffeomorphisms, Funct. Anal. Appl., 11 (1977), 229-230.

[7] R. S. Ismagilov, Unitary representations of groups of measure-preserving diffeomorphisms, Funct. Anal. Appl., 12 (1978), 226-227.

[8] A. A. Kirillov, Unitary representations of the group of diffeomorphisms and some of its subgroups, preprint IPM, No. 82 (1974).

[9] Yu. A, Neretin, The complementary series of representations of the group of diffeomorphisms of the circle, Russian Math. Surveys, 37-2 (1982), 229-230.

[10] Yu. A. Neretin, Unitary representations with highest weight of the group of diffeomorphisms of a circle, Funct. Anal. Appl., 17 (1983), 235-237.

[11] G. Segal, Unitary representations of some infinite dimensional groups, Commun. Math. Phys., 80 (1981), 301-342.

[12] A. M. Vershik, I. M. Gelfand and M. I. Graev, Representations of the group of diffeomorphisms, Russian Math. Surveys 30-6 (1975), 1-50.

[13] W. Arveson, An invitation to C^*-algebra, Chapter 3, Graduate Texts in Math. 39, Springer-Verlag, 1976.

[14] G. Choquet, Lectures on analysis, Vol. II, Chapter 7, Benjamin, 1969.

[15] A. A. Kirillov, Dynamical systems, factors and representations of groups, Russian Math. Surveys, 22-5 (1967), 63-75.

[16] K. Matthes, J. Kerstan and J. Mecke, Infinitely divisible point processes, John Wiley & Sons, 1978.

[17] N. Obata, Certain unitary representations of the infinite symmetric group I, II, to appear in Nagoya Math. J.

[18] C. Preston, Random fields, Lecture Notes in Math., 534, Springer-Verlag, 1976.

[19] E.T.Shavgulidze, An example of a measure quasi-invariant under the action of the diffeomorphism group of the circle, Funct. Anal.Appl., 12 (1978), 203-207.

[20] Y.Yamasaki, Measures on infinite dimensional spaces, Series in Pure Math., 5, World Scientific, 1985.